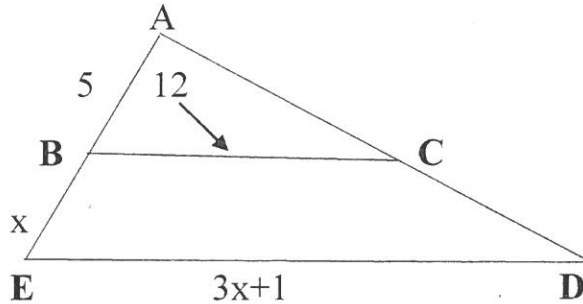


5 Geometric Similarities Oct 2015 (You may use calculators)

3 pts 1. In the diagram, segment BC is parallel to segment ED. Segment AB = 5, segment BC = 12, BE = x and DE = 3x + 1. Find the length of segment AE.



Ans. _____

4 pts 2. The areas of two similar triangles are 96 and 6 respectively. If the perimeter of the larger triangle is 48, what is the perimeter of the smaller triangle?

Ans. _____

5 pts 3. In right triangle ABC, the altitude to the hypotenuse AB is drawn intersecting the hypotenuse at point E. If $BC = 3\sqrt{5}$ and $BE = 5$, find the length of segment AC.

Ans. _____

6 Team Oct 2015 (You may use calculators)

3 pts 1. Let the set of all real solutions to $|x|^2 - 6|x| + 8 = 0$ be S . What is the sum of the absolute value of the elements of S .

(1) Ans. _____ 3 pts

3 pts 2. Find the smallest positive integer greater than one which satisfies the following

$$|x^2 - 2x + 1| > x$$

(2) Ans. _____ 3 pts

3 pts 3. If $\#$ is defined for all positive integers a and b by $a \# b = 6ab - b^2$, then find

The value of $(5 \# 2) \# (3 \# 1)$.

(4) Ans. _____ 3 pts

4 pts 4. XYZ is a three-digit natural number where X , Y , and Z are the digits. $X \geq 1$, $Y = X + 1$, and $Z = Y + 1$. The greatest prime factor of XYZ is 7. Find XYZ .

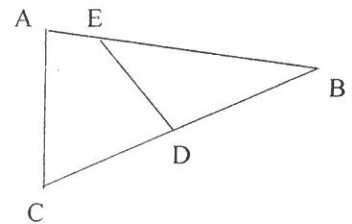
(3) Ans. _____ 4 pts

4 pts 5. How many 3-digit prime numbers are there? Hint: the number of primes less than 1000 equals the number of hours in a week.

(5) Ans. _____ 4 pts

4 pts 6. In the figure $BE = 12$, $BD = 10$, $AE = 3$, $\angle A \cong \angle BDE$, and $\overline{DE} \cong \overline{CD}$. Find AC .

(6) Ans. _____ 4 pts



5 pts 7. Each of 2 similarly-shaped rectangular prisms have 2 edges in a ratio of 2 to 3. The volume of the larger is 1944 and the smaller is $24x$. All edges are of integral length and the ratio of any two corresponding sides of the two prisms is 1 to 3. Find the area of the largest possible face of the larger prism.

(7) Ans. _____ 5 pts

5 pts 8. For how many unique pairs (x, y) of positive integers does $x^2 - y^2 = 945$?

(8) Ans. _____ 5 pts

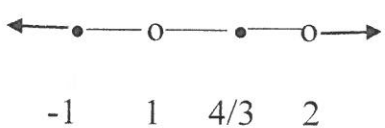
5 pts 9. Let H be a 3×3 matrix with determinant of 2. Suppose H is multiplied by 2, then transposed, then the first and third rows are interchanged, and lastly the resulting matrix is inverted. What is the determinant of the new matrix?

(9) Ans. _____ 5 pts

Solutions – Arithmetic with “*” Operations

1. $(4 * -7) = 7(4) - 4(-7) = 56$. $(56 * -2) = 7(56) - 4(-2) = 400$. **Ans. 400**
2. Let $x =$ price of can of Pepsi. Then $.5 + x/2 = x$. $x = 1$. 2000 cans cost **Ans. \$2000**
3. $2 * 1 = 4 - 3(2)(1) + 1 = -1 \rightarrow 4 * -1 = 16 - 3(4)(-1) + 1 = 29 \rightarrow$
 $5 * 29 = 25 - 3(5)(29) + 29^2 = 25 - 435 + 841 = 431$.
 $1 * 2 = 1 - 3(1)(2) + 4 = -1 \rightarrow 5 * -1 = 25 - 3(5)(-1) + 1 = 41 \rightarrow$
 $4 * 41 = 16 - 3(4)(41) + 41^2 = 16 - 492 + 1681 = 1205$. $431 + 1205 =$ **Ans. 1636**

Inequalities and Absolute Values

1. $\frac{|3-4|+4}{10} = \frac{5}{10} = \frac{1}{2}$. **Ans. $\frac{1}{2}$**
2. $-7 \leq \frac{4x-8}{9} \leq 7 \implies -63 \leq 4x-8 \leq 63 \implies -55 \leq 4x \leq 71 \implies \frac{-55}{4} \leq x \leq \frac{71}{4}$. $a + b =$ **Ans. 4**
3. Critical points are at 2 and 1 and where
- $$\frac{3x}{2x-4} = \frac{1}{1-x} \implies 3x - 3x^2 = 2x - 4 \implies 3x^2 - x - 4 = 0 \implies (3x-4)(x+1) = 0, x = 4/3 \text{ or } -1.$$
-  Plugging in interval points: $-2 \rightarrow \frac{-6}{-8} \leq \frac{1}{3}$, no; $0 \rightarrow 0 \leq 1$, yes;
 $-1 \quad 1 \quad 4/3 \quad 2 \quad 7/6 \rightarrow \frac{3.5}{-5/3} \leq -6$, no; $5/3 \rightarrow \frac{5}{-2/3} \leq \frac{1}{-2/3}$, yes.
 $3 \rightarrow \frac{9}{2} \leq \frac{1}{-2}$, no. **Ans. $-1 \leq x < 1$ or $4/3 \leq x < 2$**

Matrices, Determinants and Linear Systems

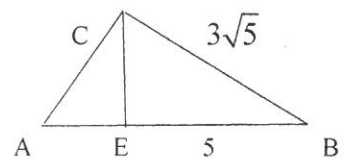
1. $A = 4$. $A - B = 10 \rightarrow 4 - B = 10$, so $B = -6$. $B + 3C = 18 \rightarrow -6 + 3C = 18$, so $C = 8$.
 $C - 2D = 0 \rightarrow 8 = 2D$, so $D = 4$. $A + B + C + D = 4 + (-6) + 8 + 4 = 10$. **Ans. 10**
2. Let $a = 1/x$ and $b = 1/y$, then (1) $a + 2b = 6$ and (2) $4a + 9b = 5$. (2) $-4(1)$: $b = -19$.
 In (1): $a + 2(-19) = 6$, so $a = 44$. $1/x - 1/y = a - b = 44 - (-19) = 63$. **Ans. 63**
3. $|A| = 10y + 9x - 10x - 27y = -71$, or (1) $-x - 17y = -71$. In multiplying the matrices:
 Third row by column yields $3x + 6y = 33$ or (2) $x + 2y = 11$. (1) + (2): $-15y = -60$, so $y = 4$.
 In (2): $x + 2(4) = 11$, so $x = 3$. $xy = (3)(4) = 12$. **Ans. 12**

Number Theory

1. If x is the first integer, then $x + 1$ and $x + 2$ are the next two. Adding makes $3x + 3$, which makes $3(x + 1)$. 3 is the factor. **Ans. 3**
2. Any number of the form n^2 has 3 factors, thus the perfect squares $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 = 285$. n^4 or n^2m^2 also have an odd number of factors. But these that are less than 100 are already in the sum. **Ans. 285**
3. Let $x =$ the smaller of the consecutive integers that make n and $n + 61$. Then $(x + 1)^2 - x^2 = 61 \rightarrow x^2 + 2x + 1 - x^2 = 61 \rightarrow 2x = 60$, so $x = 30$ and $n = 900$. Let y be the smaller of the consecutive integers that make m and $m + 61$. Then $(m + 1)^3 - m^3 = 61 \rightarrow m^3 + 3m^2 + 3m + 1 - m^3 = 61 \rightarrow m^2 + m - 20 = 0 \rightarrow (m - 4)(m + 5) = 0$. So $m = 4$ and $m = 64$. $n - m = 900 - 64 = 836$. **Ans. 836**

Geometric Similarities

1. $\frac{BC}{DE} = \frac{AB}{AE} \rightarrow \frac{12}{3x+1} = \frac{5}{x+5} \rightarrow 12x + 60 = 15x + 5 \Rightarrow 55 = 3x$, so $x = 55/3$. $AE = 5 + 55/3 = 70/3$ or $23\frac{1}{3}$. **Ans. 70/3 or 23 $\frac{1}{3}$**
2. Area ratio: $\frac{96}{6} = \frac{16}{1}$. So ratio of perimeters is 4:1. Therefore smaller is 12. **Ans. 12**
3. Use figure at right. $\frac{BE}{BC} = \frac{BC}{AB}$ or $\frac{5}{3\sqrt{5}} = \frac{3\sqrt{5}}{AB} \rightarrow 5AB = 45$, so $AB = 9$.
- Now $\frac{AE}{AC} = \frac{AC}{AB}$ or $\frac{4}{AC} = \frac{AC}{9} \rightarrow (AC)^2 = 36$, so $AC = 6$. **Ans. 6**



Team

1. $|x|^2 - 6|x| + 8 = 0 \rightarrow (|x| - 4)(|x| - 2) = 0$. So $|x| - 4 = 0 \rightarrow x = 4$ or -4 ; or $|x| - 2 = 0 \rightarrow x = 2$ or -2 .
The sum of absolute values is $4 + 4 + 2 + 2 = 12$. **Ans. 12**
2. By simply plugging in: $2 \rightarrow |4 - 4 + 1| > 2$, no. $3 \rightarrow |9 - 6 + 1| > 3$, yes. **Ans. 3**
3. $5 \# 2 = 6(5)(2) - 2^2 = 60 - 4 = 56$. $3 \# 1 = 6(3)(1) - 1 = 18 - 1 = 17$.
 $56 \# 17 = 6(56)(17) - 17 = 5712 - 289 = 5423$. **Ans. 5423**
4. There are seven numbers to try: 123, 234, 345, 456, 567, 678, 789. Each is divisible by 3, making 41, 78, 115, 152, 189, 226, 263. GPF: 41, 13, 23, 19, 7, 11, 263. **Ans. 567**

5. There are $7(24) = 168$ primes less than 1000. There are 25 primes less than 100.

$$168 - 25 = 143.$$

Ans. 143

6. Since the triangles are similar: $\frac{BE}{BC} = \frac{BD}{AB}$ or $\frac{12}{BC} = \frac{10}{15} \rightarrow 10BC = 180$, so $BC = 18$.

Thus $CD = 8 = DE$. $\frac{DE}{AC} = \frac{BD}{AB} \rightarrow \frac{8}{AC} = \frac{10}{15}$, so $AC = 12$.

Ans. 12

7. Volume ratio = $(1/3)^3 = 1/27$. Thus $\frac{1}{27} = \frac{24x}{1947} \rightarrow x = 3$. The two-edge ratio of 2:3 for each solid could be 2:3:12 for the smaller's volume to be $24(3)$. Thus the larger would have to have edges of 6, 9, and 36, with largest face area of $9(36) = 324$. The three edges of the smaller could be 3, 4 and 6, which would produce a 2:3 ratio of edges and thus the larger's edges would be 9, 12 and 18, with largest face area $12(18) = 216$. So largest area = 324.

Ans. 324

8. Since $x^2 - y^2 = (x + y)(x - y)$, it is necessary to find all possible products of two positive integers equal to 945. $945 = 3^3(5)7$, so it has 16 factors. These may be grouped into 8 pairs, each with a product of 945: $945(1)$, $315(3)$, $189(5)$, $135(7)$, $105(9)$, $63(15)$, $45(21)$, $35(27)$. In each case, the sum of the factors is $2x$ and the difference is $2y$. So $945 = 473^2 - 472^2 = 159^2 - 156^2 = 97^2 - 92^2 = 71^2 - 64^2 = 57^2 - 48^2 = 39^2 - 24^2 = 33^2 - 12^2 = 31^2 - 4^2$.

Ans. 8

9. By multiplying the matrix by 2 increases the determinant by a factor of 8; transposing it does nothing, interchanging two rows negates the determinant, and taking the inverse takes the reciprocal of the determinant. Hence, the resulting determinant is $-1/16$.

Ans. $-1/16$

Answer Sheet – Oct 2015

Arithmetic with Star Operations

1. 400
2. \$2000 or 2000
3. 1636

Team

1. 12
2. 3
3. 5423

4. 567

5. 143

Inequalities and Absolute Values

1. $1/2$
2. 4
3. $-1 \leq x < 1$ or $4/3 \leq x < 2$

6. 12

7. 324

8. 8

9. $-1/16$

Matrices, Determinants and Systems

1. 10
2. 63
3. 12

Number Theory

1. 3
2. 285
3. 836

Geometric Similarities

1. $70/3$ or $23\frac{1}{3}$ or 23.33...
2. 12
3. 6