November 2000

Solutions – Arithmetic with Ratio and Proportion

1. 1/3 to ¾ = 4/12 to 9/12. E is 8/12 which equals 2/3. Ans. 2/3

2. The ratio of females to males is (1) $\frac{f}{m}=\frac{7}{4}$. On that day (2) $\frac{f-3}{m-12}=\frac{5}{2}$. In (2): $2f-6=5m-60$ or $2f=5m-54$ ➔ (3) $4f=10m-108$. In (1): $4f=7m$. Subbing this into (3): $7m=10m-108$, thus $m=36$. $4f=7(36)$, thus $f=63$. There are 99 in the band. On this day there were 15 absent, leaving 84. Ans. 84

3. I grow at ¼ ft/yr. My tree grows at 1 ½ ft/yr. $3+1\frac{1}{2}x=4\frac{1}{2}+\frac{1}{4}x$ ➔ $1\frac{1}{4}x=1\frac{1}{2}$. $\frac{4}{5}\cdot\frac{5}{4}x=\frac{3}{2}\cdot\frac{4}{5}=\frac{6}{5}$. Thus $x=1.2$. 1.2 yrs added to 11 yrs = 12.2 yrs. ~~Ans. 1.2 yrs~~

Series and Sequences

1. $241=7+(n-1)13$ ➔ $234=(n-1)13$ ➔ $18=n-1$, thus $n=19$. Ans. 19

2. (1): $(30+x)r=60+x$ ➔ $30r+xr=60+x$
(2): $(60+x)r=100+x$ ➔ $60r+xr=100+x$
(2) – (1): $30r=40$. Thus $r=4/3$.

In (1): $(30+x)4/3=60+x$
$120+4x=180+3x$
$x=60$. Ans. 60

3. The sum of the original series is $\frac{a}{1-r}=2$, or (1) $a=2(1-r)$.

The sum of the cubes is $\frac{a^3}{1-r^3}=24$, or (2) $a^3=24(1-r^3)$.

Cubing (1): $a^3=8(1-r)^3$. Thus $24(1-r^3)=8(1-r)^3$

$24(1-r)(1+2r+r^2)=8(1-r)^3$

$3+6r+3r^2=1-2r+r^2$

$2r^2+5r+2=0$ ➔ $(2r+1)(r+2)=0$

r must be – ½ since $r^2<1$. In (1): $a=2(1+½)=3$. Ans. 3

Counting Principles and Binomial Theorem

1. There are 8(10) or 80 possible outcomes considering the first and last digits only. There is only 1 favorable outcome starting with 9 and ending with 0. Ans. 1/80

2. The term: $a^n(a^{-1/2})^{7-n}$ $n+½n-7/2=-½$ ➔ $3/2\,n=3$, so $n=2$. $\binom{7}{2}=21$. Ans. 21

3. The number of ways of picking two or more senior partners is $\binom{4}{2}+\binom{4}{3}+\binom{4}{4}=$ $6+4+1=11$. The number of ways to pick none or 1 of 3 junior partners is 4. Ans. 44

Areas and Volumes

1. The shaded square is ½ of the area of the isosceles right triangle which is one fourth of the area of the square. So 'tis 1/8 of the square. Ans. 1/8

2. Each of the sides of the base is 2 feet or 24 inches. The altitude forms a 9-12-15 triangle with half the base and the altitude of each lateral-face triangle. The area of the 4 triangles is (15)(24(1/2)4 = 720. Ans. 720 in^2 or ~~5 ft^2~~

NOTE: A numerical answer of 720 or 5 is not acceptable since there is more than one dimension in the problem.

3. Let the radius of the base of the cone be x. Its height is 2x. The volume of the cone is $1/3\ \pi x^2(2x) = 2/3\ \pi x^3$. The volume of the cube is $(2x)^3 = 8x^3$. The ratio is: Ans. $\pi/12$

Polynomials

1. $3x^3-5x^2+2x-3\ \overline{)\ 6x^5-x^4-35x^3+40x^2-17x+33}$

$$\begin{array}{r} 2x^2+3x-8 \\ \hline 6x^5-x^4-35x^3+40x^2-17x+33 \\ \underline{6x^5-10x^4+4x^3-6x^2} \\ 9x^4-39x^3+46x^2-17x \\ \underline{9x^4-15x^3+6x^2-9x} \\ -24x^3+40x^2-8x+33 \\ \underline{-24x^3+40x^2-16x+24} \\ 8x+9 \end{array}$$

Ans. 8x + 9

2. $p(x) = 2x^3-5x^2-9x+18 = (x+2)(x-3)(2x-3) = 0$ Ans. –2, 3, 1 ½

3. $f(x) = x^3-3x^2+kx-5$ $f(2) =$

	1	-3	k	-5
2		2	-2	2k-4
	1	-1	k-2	2k-9

$f(-1) =$

	1	-3	k	-5
-1		-1	4	-k-4
	1	-4	k+4	-k-9

Since f(2) = f(-1) + 12, then
2k - 9 = -k – 9 + 12 ➔
3k = 12, k = 4.

$f(3) =$

	1	-3	4	-5
3		3	0	12
	1	0	4	7

Ans. 7

Team

1. $k = VTP = (\frac{8}{5}V)(\frac{1}{2}T)(aP) = (\frac{4}{5}a)(VTP)$. So a = 5/4, which is an increase of: Ans. 25%

2. $\frac{w}{x} = \frac{4}{3}$, thus (1) 3w = 4x. $\frac{y}{z} = \frac{3}{2}$, thus (2) 2y = 3z. $\frac{z}{x} = \frac{1}{6}$, thus (3) 6z = x. Subbing (3) into (1): 3w = 4(6z) = 24z. In (2): 2y = 3z, so 16y = 24z, and thus 3w = 16y. Dividing both sides by 16y: $\frac{w}{y} = \frac{16}{3}$.
Ans. 16:3 or 16/3

3. Moving the paper around does not change the volume. Converting 10 ft to inches and multiplying: 120(11)(8 ½) = 11,220 cu in
Ans. 11,220 cu in

4. The 3-4-5 Δ is the smallest with sides in an arithmetic sequence. Multiples will also be. The 15-20-25 Δ will yield the least possible area of 150.
Ans. 150

5. The sum of the first 24 days is 24(a + a + 23(4))/2 = 24a + 1104.
The sum of the 25th to the 42nd days is 9(a + 96 + a + 4(41) = 18a + 2340.
Thus 24a + 1104 = 18a + 2340, 6a = 1236, so a = 206.
The sum of all 42 days is 21(206 + 370) = 21(576) = 12,096.
Ans. 12.096

6. The sum of x integers is $\frac{x(x+1)}{2}$. Setting up to find a pattern:

integer x	sum of integers from 1 to x	sum of their cubes	fractional part
2	3	9	1/3
3	6	36	1/6
4	10	100	1/10
5	15	225	1/15

As you can see, the denominator of the fraction increases by 1 each time or the next integer. So for 6 the denominator would be 21 and for x = 7 it would be 28. Ans. 7

7. If g(x) = p(x-2) = $x^3 + 3x^2 - 2x - 1$, then g(x+3) = p(x+1) = $(x+3)^3 + 3(x+3)^2 - 2(x+3) - 1$
or $(x^3 + 9x^2 + 27x + 27) + 3(x^2 + 6x + 9) - 2(x+3) - 1 = x^3 + 12x^2 + 43x + 47$. Ans. 103

8. The driver's seat can only be taken by two possible students. That leaves 4 students to sit in 5 seats. The first student left has 5 choices as to where to sit. The next has 4 choices, then 3, then 2. The product $2 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 240$.
Ans. 240

9. Since the polynomial is entirely of even powers of x, the roots are additive inverses of each other. That is if r is a root, then –r is also a root. Since the roots are in arithmetic progression, then 3r and –3r are the other two roots, where 2r is the common difference between terms. Thus $(x-r)(x+r)(x-3r)(x+3r) = (x^2 - r^2)(x^2 - 9r^2) = x^4 - 10r^2x^2 + 9r^4$ must be equal to $x^4 - (3m+2)x^2 + m^2$. Therefore (1) $10r^2 = 3m + 2$ and (2) $9r^4 = m^2$. In (2): $m = \pm 3r^2$.

Plugging $3r^2$ into (1): $10r^2 = 3(3r^2) + 2 \rightarrow r^2 = 2$. Plugging this back into (1): $10(2) = 3m + 2$ or $m = 6$.

Plugging $-3r^2$ into (1): $10r^2 = 3(-3r^2) + 2 \rightarrow 19r^2 = 2$, so $r^2 = 2/19$. Plugging this back into (1): $10(2/19) = 3m + 2 \rightarrow 20/19 = 3m + 2$, or $-18/19 = 3m$. Thus $m = -6/19$.

Ans. 6 or $-6/19$

Answer Sheet

Arithmetic with Ratio and Proportion
1. 2/3
2. 84
3. 12.2 or 12.2 yrs

Series and Sequences
1. 19
2. 60
3. 3

Counting Principles and Binomial Theorem
1. 1/80
2. 21
3. 44

Areas and Volumes
1. 1/8 or 1:8
2. 720 sq in or 5 sq ft
3. $\pi/12$

Polynomials
1. $8x + 9$
2. 3, -2, 1 ½
3. 7

Team
1. 25 or 25%
2. 16/3 or 16:3
3. 11,220 cu in
4. 150 or 150 sq units
5. 12,096
6. 7
7. 103
8. 240
9. 6 or -6/19

Meet 2 Nov. 2000

VI Team

3 pts 1. Volume, Temperature, and Pressure are related by the equation VTP = k, where k is a constant. If the volume increases by 60% while the temperature decreases by 50%, by what percent does the pressure increase?

Ans. _____

3 pts 2. The ratio of w to x is 4:3, of y to z is 3:2, of z to x is 1:6. What is the ratio of w to y?

Ans. _____

3 pts 3. I have a 10-foot high stack of papers (8 ½ in. by 11 in.) on my desk. If I take ½ of them and form another stack, ¼ of my desk will be covered with papers. If I then divide these stacks into equal stacks that cover my desk, what will be the volume of papers in cubic inches covering my desk?

Ans. _____

4 pts 4. A right triangle with area ≥ 100 square units has integer side lengths that form an arithmetic sequence. Find the least possible area for the triangle in square units.

Ans. _____

4 pts 5. A postal employee delivered mail daily for 42 days, each day delivering 4 more letters than on the previous day. The total delivery for the first 24 days of the period was the same as that for the last 18 days. How many letters did the employee deliver during the whole 42-day period?

Ans. _____

4 pts 6. ~~M~~ m is a positive integer. The sum of all the integers from 1 through m is one twenty-eighth the sum of all the cubes of the integers from 1 through m. Find m.

Ans. _____

5 pts 7. $p(x-2) = x^3 + 3x^2 - 2x - 1$. Find a + b + c + d, if $p(x+1) = ax^3 + bx^2 + cx + d$.

Ans. _____

5 pts 8. 5 students plan to go on a "joy" ride in a sedan which seats 3 in the front seat (including the driver), and three in the back seat. If only 2 can drive, how many possible seating arrangements can be made for the ride?

Ans. _____

5 pts 9. Determine m so that the equation in x, $x^4 - (3m+2)x^2 + m^2 = 0$, has four real roots in arithmetic progression.

Ans. _____