Meet #4 March 2002

Solutions – Arithmetic with Literal Equations

1. $I = \frac{.05p + p}{R}$ ➔ $IR = 1.05p = \frac{21p}{20}$. Thus $p = 20IR/21$ Ans. $p = \frac{20IR}{21}$

2. $\frac{1}{2} = \frac{\frac{A}{B}+\frac{C}{D}}{\frac{E}{F}}$ ➔ $\frac{E}{2F} = \frac{A}{B}+\frac{C}{D}$ ➔ $\frac{E}{2F}-\frac{A}{B}=\frac{C}{D}$ ➔ $\frac{BE-2AF}{2BF} = \frac{C}{D}$ ➔ $\frac{D}{C} = \frac{2BF}{BE-2AF}$

Therefore $D = \frac{2BCF}{BE-2AF}$ Ans. $\frac{2BCF}{BE-2AF}$

3. $(4x-y)^2 = x^2 + (3x+y)^2$ ➔ $16x^2 - 8xy + y^2 = x^2 + 9x^2 + 6xy + y^2$ or $6x^2 - 14xy = 0$. Factoring: $2x(3x-7y) = 0$. $3x = 7y$ ➔ $\frac{x}{y} = \frac{7}{3}$. The ratio of x to y is 7 to 3. Ans. 7 to 3

Trigonometric Equations and Identities

1. $\sin^2 x + \cos^2 x + \tan^2 x + \sec^2 x$ ➔ $1 + \tan^2 x + \sec^2 x = \sec^2 x + \sec^2 x$. Ans. $2\sec^2 x$

2. $\cos^2 x - \sin x \cos x = 0$ ➔ $\cos x(\cos x - \sin x) = 0$ ➔ Either (1) $\cos x = 0$ or (2) $\cos x - \sin x = 0$. In (1) $x = 90°$ or $270°$. In (2) $\cos x = \sin x$ or dividing both sides by $\cos x$: $\tan x = 1$, thus $x = 45°$ or $225°$. Ans. 45°, 90°, 225°, 270°

Algebraic Fractions and Factoring

1. Multiplying top and bottom of fraction by (a-b)(a+b): $\frac{(a-b)^2+(a+b)^2}{a(a+b)-b(a-b)} =$

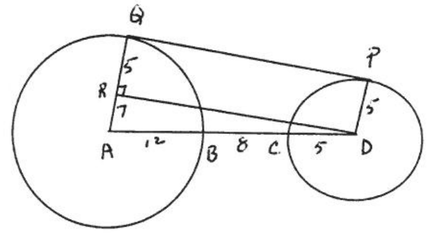
$\frac{a^2-2ab+b^2+a^2+2ab+b^2}{a^2+ab-ab+b^2} = \frac{2a^2+2b^2}{a^2+b^2} = 2$ Ans. 2

2. $\frac{1}{3-x}+\frac{5}{x+1}=\frac{8}{x^2-2x-3}$ ➔ $\frac{1}{x-3}-\frac{5}{x+1}=\frac{-8}{(x-3)(x+1)}$ $= (x+1)-5(x-3) = -8$. Therefore: $-4x + 16 = -8$ ➔ $-4x = -24$ ➔ $x = 6$. Ans. 6

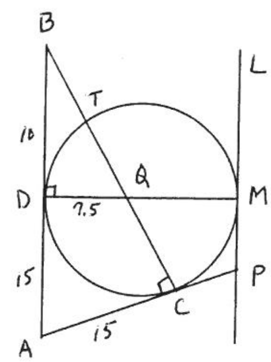
3. $\frac{x-3}{x-1}-\frac{x+1}{x+2}=\frac{x-5}{x-2}$ ➔ $(x-3)(x^2-4) - (x+1)(x^2-3x+2) = (x-5)(x^2+x-2) =$
$x^3 - 3x^2 - 4x + 12 - (x^3 - 2x^2 - 3x + 2) = x^3 - 4x^2 - 7x + 10$
$-x^2 - 3x + 10 = x^3 - 4x^2 - 7x + 10$ ➔ $0 = x^3 - 3x^2 - 4x = x(x^2 - 3x - 4)$. Factoring this: $x(x-4)(x+1) = 0$
Ans. 0, 4, -1

Circles and Spheres

1. $9(16) = DC^2 \Rightarrow DC = 12$ Ans. 12



2. Connecting A to Q and D to P, makes right angles at P and Q. Drawing segment DR parallel to PQ, makes rectangle RDPQ. Since CD = 5, then PD =5, and thus RQ is 5. Since AQ = 12, Then AR = 7. In right triangle ARD, hypotenuse AD = 25, thus RD = 24 and so does PQ. Ans. 24



3. Since $\overline{BC} \perp \overline{AP}$, then $\overline{BC}$ passes through the center. Let D be the point of tangency of $\overline{AB}$, and M be the point of tangency of $\overline{PL}$. Connecting D to M results in perpendiculars at these points. AC = 15, then AD = 15, since tangents to a circle from the same point are congruent. Because AB = 25, BD = 10. Δ's BDQ and BCA are similar. Thus $\frac{BD}{DQ} = \frac{BC}{AC} \Rightarrow \frac{10}{DQ} = \frac{20}{15}$. Thus DQ = 7.5. Thus CT = 15 and BT = 5. Ans. 5

Conics

1. $(x^2 - 16x + 64) + 4(y^2 - 6y + 9) = -84 + 64 + 36 = 16$

$\frac{(x-8)^2}{16} + \frac{(y-3)^2}{4} = 1$, Center (8,3), vertices $(8 \pm 4, 3)$ Ans. (12,3), (4,3)

2. Since the focus is further away from the center than the vertex, the conic must be a hyperbola of the form $\frac{y^2}{16} - \frac{x^2}{b^2} = 1$. Since $16 + b^2 = (2\sqrt{13})^2$, then $b^2 = 52 - 16$. b = 6.

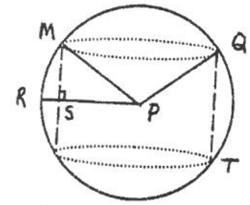
Thus the equation is $\frac{y^2}{16} - \frac{x^2}{6^2} = 1$ Ans. $\frac{y^2}{16} - \frac{x^2}{36} = 1$

3. Plugging (12,5) into $7x - 24y = c$: $84 - 120 = -36$. Thus the distance between the two parallel lines now made is $\frac{89-(-36)}{\sqrt{7^2 + 24^2}} = \frac{125}{25} = 5$. Thus the radius of the circle is 5.

Ans. $(x-12)^2 + (y-5)^2 = 25$

Team

1. The center of the other circle must be at (4,4). Ans. $(x-4)^2 + (y-4)^2 = 8$



2. Connecting P to M, the triangle PSM is a 45-45-90Δ. MS = $10\sqrt{2}$ and so is PS. Therefore RS = $20 - 10\sqrt{2}$. Ans. $20 - 10\sqrt{2}$.

3. The area of the 8-15-17Δ is ½ (8)(15) = 60. The area can also be figured by using the formula A = ½ ap, where a is the apothem or in this case the radius of the inscribed circle and p is the perimeter. So 60 = ½ a(40), thus a = 3.

Ans. 3

4. $2\cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\cos^2\theta + 1$

$2\cos^2\theta\,(\cos^2\theta + \sin^2\theta) - 2\cos^2\theta + 1$

$2\cos^2\theta - 2\cos^2\theta + 1 \Rightarrow 1$

Ans. 1

5. $a = b - dc^m \Rightarrow a - b = -dc^m \Rightarrow \frac{b-a}{d} = c^m \Rightarrow \log_c \frac{b-a}{d} = \log_c c^m = m$. Ans. $\log_c \frac{b-a}{d}$

6. One asymptote is $2x - 5y = -23$. The other is $y = -\frac{2}{5}x + 1\frac{2}{5}$ or $2x + 5y = 7$. Adding these to find where they meet: $4x = -16$, so $x = -4$. Plugging this back into the first equation: $2(-4) - 5y = -23 \Rightarrow -5y = -15$ or $y = 3$. Thus the center of the hyperbola is (-4,3) With the vertex at (-4,1), the semi-transverse axis is 2. Since the vertices are on a line parallel to the y-axis, then the equation so far is $\frac{(y-3)^2}{4} - \frac{(x+4)^2}{b^2} = 1$. Since the asymptotes have a slope of 2/5, then the semi-conjugate axis (b) is 5.

Ans. $\frac{(y-3)^2}{4} - \frac{(x+4)^2}{25} = 1$

7. $4\sqrt{\frac{3-x}{3+x}} - \sqrt{\frac{3+x}{3-x}} = \sqrt{2} \Rightarrow 4\sqrt{\frac{(3-x)(3+x)}{(3+x)(3+x)}} - \sqrt{\frac{(3+x)(3-x)}{(3-x)(3+x)}} = \sqrt{2}$

$\frac{4\sqrt{9-x^2}}{3+x} - \frac{\sqrt{9-x^2}}{3-x} = \sqrt{2} \Rightarrow 4(3-x)\sqrt{9-x^2} - (3+x)\sqrt{9-x^2} = \sqrt{2}\,(9-x^2)$

$12 - 4x - (3 + x) = \sqrt{2}\,\sqrt{9-x^2} \Rightarrow 9 - 5x = \sqrt{18-2x^2} \Rightarrow 81 - 90x + 25x^2 = 18 - 2x^2 \Rightarrow$ $27x^2 - 90x + 63 = 0 \Rightarrow 3x^2 - 10x + 7 = 0 \Rightarrow (3x-7)(x-1) = 0$. $x = 1$ or 7/3. But 7/3 doesn't work.

Ans. 1

8. (1) $\cos A + \cos B = 1$, (2) $\sin A + \sin B = \sqrt{3}$. Squaring each and adding together: $\cos^2 A + 2\cos A\cos B + \cos^2 B = 1$, $\sin^2 A + 2\sin A\sin B + \sin^2 B = 3$ $\cos^2 A + \sin^2 A + 2\cos A\cos B + 2\sin A\cos B + \cos^2 B + \sin^2 B = 4$. This simplifies to: $\cos A\cos B + \sin A\sin B = 1 \Rightarrow \cos(A - B) = 1$. Thus $A - B = 0°$ or $360°$. The smallest possible value is 360°.

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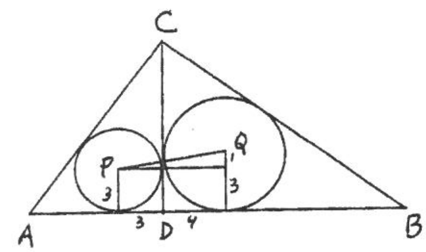
Ans. 360°

9. In Δ ACB since AC = 15 and BC = 20, then AB = 25. To determine the length of altitude DC: DC(AB) = AC(BC) or DC(25) = 15(20) and thus DC = 12. In right Δ ADC, AC = 15, DC = 12 and therefore AD = 9 and thus BD = 16. In Δ ADC, the radius of the circle is Area = ½ ap ➔ 54 = ½ a(36) or a = 3. In Δ BDC, 96 = ½ a(48) or a = 4. Dropping the perpendicular from P to AD at M and from Q to BD at N. Then MN = 7. Connecting a parallel line segment from P to QN at R, makes right Δ PQR. Since RN = 3, then RQ = 1, and PR = 7. Therefore PQ = $\sqrt{50}$ or $5\sqrt{2}$. Ans. $5\sqrt{2}$



Answer Sheet

Arithmetic with Literal Equations

1. $\frac{20IR}{21}$ or $p = \frac{20IR}{21}$
2. $\frac{2BCF}{BE - 2AF}$
3. 7 to 3 or 7:3 or 7/3

Trigonometric Equations and Identities

1. $2\sec^2 x$
2. $90°, 270°, 45°, 225°$
3. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Algebraic Fractions and Factoring

1. 2
2. 6
3. 0, 4, or –1

Circles and Spheres

1. 12
2. 24
3. 5

Conics

1. (12,3) and (4,3)
2. $\frac{y^2}{16} - \frac{x^2}{36} = 1$
3. $(x - 12)^2 + (y - 5)^2 = 25$

Team

1. $(x - 4)^2 + (y - 4)^2 = 8$
2. $20 - 10\sqrt{2}$
3. 3
4. 1
5. $\log_c \frac{b - a}{d}$
6. $\frac{(y-3)^2}{4} - \frac{(x+4)^2}{25} = 1$
7. 1
8. $360°$ or 2π
9. $5\sqrt{2}$

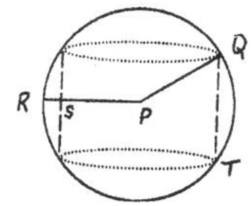
Meet #4 March 2002

VI Team

3 pts 1. Two congruent circles are externally tangent at (2,2). The equation of one is $x^2 + y^2 = 8$. Write the equation of the other circle.

Ans. ______________________

3 pts 2. The right circular cylinder is inscribed in the sphere with center P. $QT = 20\sqrt{2}$, $PR = 20$. Find the length of RS in simplest form.



Ans. ______________________

3 pts 3. Find the radius of the inscribed circle of an 8-15-17Δ.

Ans. ______________

4 pts 4. Simplify $2\cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\cos^2\theta + 1$

Ans. ______________

4 pts 5. If $a = b - dc^m$, solve for m.

Ans. ______________________

4 pts 6. One of the asymptotes of a hyperbola is $2x - 5y = -23$. The other is $y = -\frac{2}{5}x + 1\frac{2}{5}$. Find its equation, if one of its vertices is (-4,1).

Ans. ______________________

5 pts 7. Find all value(s) of x such that

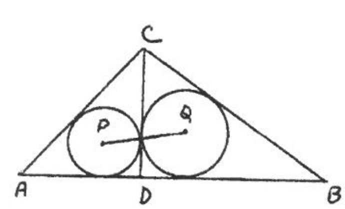
$$4\sqrt{\frac{3-x}{3+x}} - \sqrt{\frac{3+x}{3-x}} = \sqrt{2}$$

Ans. ______________________

5 pts 8. If $\cos A + \cos B = 1$ and $\sin A + \sin B = \sqrt{3}$, compute the smallest possible value of $A - B$.

Ans. ______________

5 pts 9. In triangle ABC, $\angle C$ is a right $\angle$. $\overline{CD}$ is an altitude. The circles centered at P and Q are inscribed in the triangles. If AC = 15, and BC = 20, compute the length of $\overline{PQ}$.



Ans. ______________________