Meet #5 March 2000

SOLUTIONS – Algebraic Fractions with Factoring

1. $\dfrac{12x^2 - x - 20}{12x^4 - 25x^3 + 12x^2} = \dfrac{(3x-4)(4x+5)}{x^2(3x-4)(4x-3)} = \dfrac{4x+5}{x^2(4x-3)}$ Ans. $\dfrac{4x+5}{4x^3 - 3x^2}$

2. $x^3 - x^2 - 4x + 4 = x^2(x-1) - 4(x-1) = (x^2 - 4)(x-1) = (x-2)(x+2)(x-1)$. Thus:

$$\frac{1}{x^2-3x+2} + \frac{1}{x^2-5x+6} + \frac{1}{x^3-x^2-4x+4} = \frac{1}{(x-2)(x-1)} + \frac{1}{(x-3)(x-2)} + \frac{1}{(x-2)(x+2)(x-1)}$$

The LCD = $(x-1)(x-2)(x+2)(x-3)$. Thus the numerator is: $(x+2)(x-3) + (x-1)(x+2) + (x-3)$ which is $x^2 - x - 6 + x^2 + x - 2 + x - 3 = 2x^2 + x - 11$. Ans. $\dfrac{2x^2 + x - 11}{(x-1)(x-2)(x+2)(x-3)}$

3. $\dfrac{\dfrac{1}{x-3} + \dfrac{1}{x-2}}{1 - \dfrac{2x-9}{x^2-5x+6}} = \dfrac{7}{9} \Rightarrow \dfrac{\dfrac{x-2+x-3}{(x-3)(x-2)}}{\dfrac{x^2-5x+6-2x+9}{(x-3)(x-2)}} = \dfrac{7}{9} \Rightarrow \dfrac{2x-5}{x^2-7x+15} = \dfrac{7}{9} \Rightarrow$

$18x - 45 = 7x^2 - 49x + 105 \Rightarrow 0 = 7x^2 - 67x + 150 = (7x-25)(x-6)$ Ans. 6 or $3\,\frac{4}{7}$

Trigonometric Equations and Identities

1. sin (Arccos 4/5) = 3/5. cos (arcsec 3) = 1/3. tan (arctan 2/3) = 2/3. Thus
$3/5 + 1/3 - 2/3 = 3/5 - 1/3 = \dfrac{9-5}{15} = \dfrac{4}{15}$ Ans. $\dfrac{4}{15}$

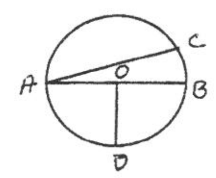
2. $\dfrac{\sin x \csc y + \cos x \sec y}{\sin x \csc y - \cos x \sec y} = \dfrac{\dfrac{\sin x}{\sin y} + \dfrac{\cos x}{\cos y}}{\dfrac{\sin x}{\sin y} - \dfrac{\cos x}{\cos y}} = \dfrac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$ Ans. $\dfrac{\sin(x+y)}{\sin(x-y)}$

3. $\cos^4\theta$ - $6\cos^2\theta$ sin $\cos^4\theta$ - $6\cos^2\theta \sin^2\theta + \sin^4\theta + \sin^4\theta = 0 \Rightarrow \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta = 4\cos^2\theta\sin^2\theta$
$(\cos^2\theta - \sin^2\theta)^2 = (2\cos\theta\sin\theta)^2 \Rightarrow (\cos 2\theta)^2 = (\sin 2\theta)^2$. Dividing both sides by $(\cos 2\theta)^2$ produces $(\tan 2\theta)^2 = 1$. Therefore $\tan 2\theta = \pm 1$, and $2\theta = \pi/4, 3\pi/4, 5\pi/4$ or $7\pi/4$. Thus $\theta = \pi/8, 3\pi/8, 5\pi/8$ or $7\pi/8$, where $0 \le \theta \le \pi$.

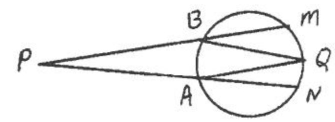
Ans. $\pi/8, 3\pi/8, 5\pi/8$ or $7\pi/8$

Circles and Spheres

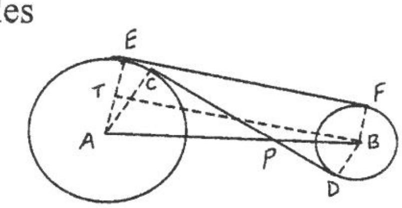
1. If AB = 6, then the circumference is 6π. Since $m\angle BAC = 15$, then $m\overset{\frown}{BC} = 30$. Since $\overline{OD} \perp \overline{AB}$ then $m\overset{\frown}{BD} = 90$, and therefore $m\overset{\frown}{CBD} = 120$. Thus $m\overset{\frown}{CAD}$ is 2/3 of the perimeter of the circle.
$2/3\,(6\pi) = 4\pi$ Ans. 4π



2. In the figure, m∠MBQ = 22, and m∠NAQ = 18. This makes m∠PAQ = 162 and m∠PBQ = 158. Adding: 158 + 162 = 340. Since there are 360° in the sum of the angles of a quadrilateral, then the sum of the measures of P and Q is 40. Ans. 40°



3. Connecting A to C and B to D makes two similar triangles such that $\frac{CP}{25} = \frac{28 - CP}{10}$ ➔ 10CP = 25(28 – CP) or 35CP = 25(28) or CP = 20. Thus PD = 8. Which means through the Pythagorean Theorem, AC = 15 and BD = 6. Now connecting F to B abd E to A and then drawing a segment parallel to FE through B to meet segment AE at T. Quadrilateral BFET is a rectangle. Since BF = 6, ET = 6 and AT = 9. Since AB = 35, then BT = $\sqrt{1225 - 81} = \sqrt{1144} = 2\sqrt{286} = 33.823$ Ans. $2\sqrt{286}$ or 33.823



Arithmetic with Literal Equations

1. $43_5 = 4(5) + 3 = 23$. $17_8 = 8 + 7 = 15$. 15 + 23 = 38. 38 = 32 + 4 + 2 Ans. 100110_2

2. $\frac{2a^2 - b^2}{a - b} = \frac{a^2 - ab + b^2}{a - b}$ ➔ $a^2 + ab - 2b^2 = 0$ ➔ (a – b)(a + 2b) = 0. Either a = b or a = -2b. But a cannot equal b.

Ans. –2b

3. $\frac{1}{2} gt^2 = \frac{1}{\sqrt{a^2 + b^2}}$ ➔ $1/4 g^2 t^4 = \frac{1}{a^2 + b^2}$ or $a^2 + b^2 = \frac{4}{g^2 t^4}$. Therefore $b^2 = \frac{4}{g^2 t^4} - a^2$.

$b^2 = \frac{4}{g^2 t^4} - \frac{a^2 g^2 t^4}{g^2 t^4}$. Thus $b = \pm\frac{\sqrt{4 - a^2 g^2 t^4}}{gt^2}$.

Ans. $b = \pm\frac{\sqrt{4 - a^2 g^2 t^4}}{gt^2}$

Conics

1. $(x^2 - 4x + 4) + (y^2 - y + \frac{1}{4}) = 4\frac{3}{4} + 4\frac{1}{4}$. Center is (2,1/2) Ans. (2,1/2)

2. With center (4,6) being tangent to x-axis and major axis parallel to the x-axis, the ellipse must have the form so far: $\frac{(x-4)^2}{a^2} + \frac{(y-6)^2}{36} = 1$. Since ecc = $\frac{\sqrt{3}}{2}$, then $\frac{f}{v} = \frac{\sqrt{3}}{2} = \frac{\sqrt{3}x}{2x}$. $(2x)^2 - (\sqrt{3}x)^2 = 6^2$ ➔ $4x^2 - 3x^2 = 36$ or $x^2 = 36$ and x = 6. v = 2x = 12.

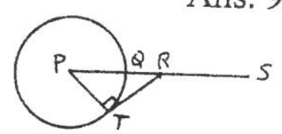
Ans. $\frac{(x-4)^2}{144} + \frac{(y-6)^2}{36} = 1$

3. (1) $x - 2y = 11$, (2) $x + 2y = -1$. Adding (1) and (2): $2x = 10$, $x = 5$. Plugging in (2): $5 + 2y = -1$ or $y = -3$. Therefore center of hyperbola is (5,-3). Since the endpoint of the conjugate axis is (5,-5), then the conjugate axis is 2 units long. If the slope of one of the asymptotes is ½ or then ½ = 2/a or a = 4 where a is the semi-horizontal axis. Since the transverse axis is horizontal, then the hyperbolic equation is: Ans. $\frac{(x-5)^2}{16} - \frac{(y+3)^2}{4} = 1$

Team

1. The center of $x^2 + y^2 = 4$ is (0,0). Its radius is 2. The center of the circle $(x-3)^2 + (y-4)^2 = R$ is (3,4). The distance between the centers is 5, so the radius of the other circle has to be 3, if the circles are to be tangent. Thus R = 9. Ans. 9

2. Connecting P to T, if PQ = 5, then PT = 5. Since R is the midpoint of PS, then PR = 6 ½. By the Pyth. Thm. $(13/2)^2 - 5^2 = RT^2$. $169/4 - 25 = 69/4 = RT^2$. Thus RT = $\sqrt{69}/2$



Ans. $\sqrt{69}/2$

3. $\sin x + \cot x = \csc x \Rightarrow \sin x + \frac{\cos x}{\sin x} = \frac{1}{\sin x} \Rightarrow \sin^2 x + \cos x = 1 \Rightarrow$ $1 - \cos^2 x + \cos x = 1 \Rightarrow \cos^2 x - \cos x = 0 \Rightarrow \cos x(\cos x - 1) = 0$. Therefore either: (1) $\cos x = 0$, which means $x = 90°$ or $270°$, but only $90°$ works. Or (2) $\cos x = 1$, which means that $x = 0°$, which cannot be used since $\csc 0°$ is undefined. Ans. $90°$

4. $$\left(1-\frac{a}{a-\frac{1}{a}}\right)\left(\frac{1}{a+\frac{1}{a-\frac{1}{a}}}\right) = \left(1-\frac{a}{\frac{a^2-1}{a}}\right)\left(\frac{1}{a+\frac{1}{\frac{a^2-1}{a}}}\right) = \left(1-\frac{a^2}{a^2-1}\right)\left(\frac{1}{a+\frac{a}{a^2-1}}\right) =$$

$$\left(\frac{a^2-1-a^2}{a^2-1}\right)\left(\frac{1}{\frac{a^3-a+a}{a^2-1}}\right) = \left(\frac{-1}{a^2-1}\right)\left(\frac{a^2-1}{a^3}\right) = -\frac{1}{a^3}$$

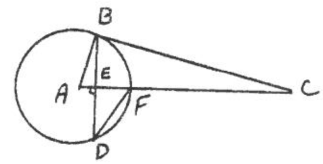
Ans. $-\frac{1}{a^3}$

5. $\frac{x-1}{x^2+3x+2} + \frac{3x+1}{2x^2+3x+1} = \frac{3x+5}{2x^2+5x+2} \Rightarrow \frac{x-1}{(x+2)(x+1)} + \frac{3x+1}{(2x+1)(x+1)} = \frac{3x+5}{(2x+1)(x+2)} \Rightarrow$ $(x-1)(2x+1) + (3x+1)(x+2) = (3x+5)(x+1) \Rightarrow 2x^2 - x - 1 + 3x^2 + 7x + 2 = 3x^2 + 8x + 5$ $2x^2 - 2x - 4 = 0$ or $x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$. $x = 2$ or -1, but $x \neq -1$. Ans. 2

6. $\cos 2\theta + \sin^2\theta - \frac{1}{4}\sin^2 2\theta = \frac{1}{4} \Rightarrow \cos^2\theta - \sin^2\theta + \sin^2\theta - \frac{1}{4}(2\sin\theta\cos\theta)^2 = \frac{1}{4}$.
$\cos^2\theta - \sin^2\theta\cos^2\theta = \frac{1}{4} \Rightarrow \cos^2\theta(1 - \sin^2\theta) = \frac{1}{4} \Rightarrow \cos^4\theta = \frac{1}{4}$. $\cos^2\theta = \pm\frac{1}{2}$. And thus $\cos\theta = \pm\frac{1}{\sqrt{2}}$. So $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

Ans. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

7. Since $\overline{AF}$ bisects $\overline{BD}$, then $\overline{AF} \perp \overline{BD}$ and Δ's AEB and ABC are similar. $\frac{AE}{7} = \frac{7}{25}$ or $AE = \frac{49}{25}$, so $EF = 7 - \frac{49}{25} = \frac{126}{25}$.



Also $\frac{BE}{7} = \frac{24}{25}$, so $BE = \frac{168}{25} = ED$. Now through the Pyth. Thm.,

$$DF^2 = \left(\frac{168}{25}\right)^2 + \left(\frac{126}{25}\right)^2 = \frac{44100}{25^2}. \quad DF = \frac{210}{25} = 8\tfrac{2}{5}.$$

Ans. $8\frac{2}{5}$

8. The area of an ellipse is $ab\pi$, where a and b are the lengths of the semi-major and semi-minor axes. $9x^2 + 16y^2 - 36x - 128y + 148 = 0$

$$9(x^2 - 4x + 4) + 16(y^2 - 8y + 16) = -148 + 36 + 256 = 144$$

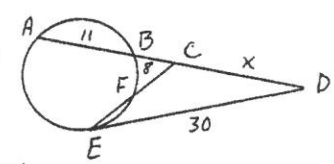
Thus $\frac{(x-2)^2}{16} + \frac{(y-9)^2}{9} = 1$, and the area of the ellipse is 12π.

The circle $x^2 + y^2 - 4x - 8y + 11 = 0$

$(x^2 - 4x + 4) + (y^2 - 8y + 16) = -11 + 4 + 16 = 9$ has an area of 9π.

Ans. 3π

9. Let CD = x. Then $30^2 = (x+8)(x+19)$ or $900 = x^2 + 27x + 152$. $x^2 + 27x - 748 = 0$. Factoring: $(x - 17)(x + 44) = 0$. So CD = 17 and so does EC. $CF \bullet CE = CB \bullet CA$ or $CF(17) = 8(19)$.
$\therefore CF = 8\frac{16}{17}$ and $FE = 17 - 8\frac{16}{17} = 8\frac{1}{17}$



Ans. $8\frac{1}{17}$

ANSWER SHEET

Algebraic Fractions and Factoring

1. $\dfrac{4x+5}{4x^3-3x^2}$
2. $\dfrac{2x^2+x-11}{(x-1)(x-2)(x+2)(x-3)}$
3. 6 or $3\frac{4}{7}$

Trigonometric Equations and Identities

1. $\dfrac{4}{15}$
2. $\dfrac{\sin(x+y)}{\sin(x-y)}$
3. $\dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}$ or ~~22 ½°, 67 ½°, 112 ½°, 157 ½°~~

Circles and Spheres

1. 4π
2. 40 or 40°
3. $2\sqrt{286}$ or 33.82

Arithmetic with Literal Equations

1. 100110
2. –2b or a = -2b
3. $\pm\dfrac{\sqrt{4-a^2g^2t^4}}{gt^2}$ or $b = \pm\dfrac{\sqrt{4-a^2g^2t^4}}{gt^2}$

Conics

1. (2, ½)
2. $\dfrac{(x-4)^2}{144}+\dfrac{(y-6)^2}{36}=1$
3. $\dfrac{(x-5)^2}{16}-\dfrac{(y+3)^2}{4}=1$

Team

1. 9 or R = 9
2. $\dfrac{\sqrt{69}}{2}$
3. 90°
4. $-\dfrac{1}{a^3}$
5. 2
6. $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ or

   45°, 135°, 225°, 315°
7. 8 ⅖
8. 3π
9. $8\frac{1}{17}$