Meet #5 2007

Solutions – Algebraic Fractions and Factoring

1. $\frac{3}{y-3}=\frac{6}{y^2-9}$ ➔ $3(y+3)=6$ ➔ $y+3=2$, so $y=-1$ **Ans. -1**

2. $\frac{C}{x-2}+\frac{D}{x+1}=\frac{6x}{x^2-x-2}$ ➔ $C(x+1)+D(x-2)=6x$ ➔ $Cx+C+Dx-2D=6x$. Thus $C+D=6$ and $C-2D=0$. Solving these $C=4$ and $D=2$. **Ans. $C=4$, $D=2$**

3. $\frac{x(x+3)(x+2)}{4-y-4x}\cdot\frac{16x^2-(y-4)^2}{(x-2)(x-4)(x-3)}=\frac{x}{4-y-4x}\cdot\frac{(4x-y+4)((4x+y-4)}{x-4}$ **Ans.** $\frac{-4x^2+xy-4x}{x-4}$

Trigonometric Equations and Identities

1. $\frac{1}{\sec x}(\tan x+\cot x)=\cos x\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)=\sin x+\frac{\cos^2 x}{\sin x}=\frac{\sin^2 x}{\sin x}+\frac{\cos^2 x}{\sin x}=\frac{1}{\sin x}$

Ans. csc x or $\frac{1}{\sin x}$

2. $\cos^4 2\theta - \sin^4 2\theta = 1$ ➔ $(\cos^2 2\theta + \sin^2 2\theta)(\cos^2 2\theta - \sin^2 2\theta) = 1$. Thus $\cos^2 2\theta - \sin^2 2\theta = 1$ or $\cos 4\theta = 1$. $4\theta = 0^o$ and will repeat every 90^o.

Ans. $90^o k \mid k \in I$ or $\frac{\pi k}{2} \mid k \in I$

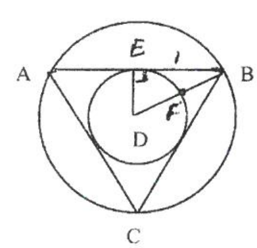
3. $\tan 105^o = \tan(60^o + 45^o) = \frac{\tan 60^o + \tan 45^o}{1-\tan 60^o \tan 45^o} = \frac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\frac{\sqrt{3}+1}{1+\sqrt{3}} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

By using the half angle formula, you get the same answer and also $-\sqrt{7+4\sqrt{3}}$

Ans. $-2-\sqrt{3}$ or $-\sqrt{7+4\sqrt{3}}$

Circles and Spheres

1. Since AB = 2 then BE = 1. Triangle BDE is a 30-60-90 triangle. So DE = $1/\sqrt{3}$ and BD = $2/\sqrt{3}$. Thus BF = BD – DE = $\sqrt{3}/3$



Ans. $\sqrt{3}/3$

2. Connecting the centers of the spheres forms a cube of side length ½ e. The length of the inner diagonal of this cube is the distance between the centers of the two spheres in question. That distance is: $d=\sqrt{(\frac{1}{2}e)^2+(\frac{1}{2}e)^2+(\frac{1}{2}e)^2}=\sqrt{\frac{3}{4}e^2}=\frac{\sqrt{3}}{2}e$ **Ans.** $\frac{\sqrt{3}}{2}e$

3. Making a triangle of centers of the three barrels creates an equilateral triangle with sides of length 2 ft. To find the center of the hoop's circle, you'd take ⅔ of the height of the triangle plus 1 foot: $\frac{2}{3}(\sqrt{3})+1$. The circumference of the hoop is $2\pi\left(\frac{2\sqrt{3}}{3}+1\right)$. This is approximately equal to 13.5 ft. **Ans. 13.5 ft.**

Conics

1. $x^2 - 4x + 4 + y^2 + 10y + 25 = 7 + 4 + 25 = 36$ **Ans. Center (2, -5), radius 6**

2. The Pythagorean triples having integral lengths and hypotenuse 25 are 7-24-25 and 15-20-25. r can be 7, 15, 20, 24. The sum is 66. **Ans. 66**

3. Since the sum of the distances is 12, if you were to make an isosceles triangle with base vertices (7, 2) and (3, 2) and connect to the endpoint of the minor axis, the sides of the triangle would be 6 (which is also the length of the semi-major axis) and through the Pythagorean theorem the altitude of the triangle (the length of the semi-minor axis) would be $\sqrt{32}$. Thus the ellipse equation: $\frac{(x-5)^2}{36}-\frac{(y-2)^2}{32}=1$. **Ans.** $\frac{(x-5)^2}{36}-\frac{(y-2)^2}{32}=1$

Alternate Solution: If you were to choose the y-value of 2 to be the coordinates of one of the vertices of the ellipse, then the distance to the closer focus is n and the distance to the further focus is $n + 4$. These added should equal 12: $n + n + 4 = 12$, thus $n = 4$. Thus the semi-major axis is 6, and using the Pythagorean Theorem the semi-minor will be $\sqrt{32}$.

Arithmetic with Statistics

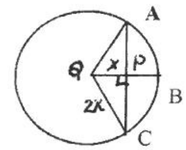
1. $6(37) + 5(42) + 4(53) + 3(49) = 791$. $791 \div 18 = 44$ the mean (rounded). The mode is 37 and the median is 42. The sum of these is 123. **Ans. 123**

2. Since the range is 10 and the maximum is 24, then the minimum is 14 If the mode is 21 then their must be at least two 21's. If the mean is 19, then the sum of all the values is $6(19) = 114$. If the median is 20 and there are an even number of terms, then 19 has to be the value in order below 21, thus $24 + 21 + 21 + 19 + n + 14 = 114$, $n = 15$.

Ans. 14, 15, 19, 21, 21, 24

3. If n numbers have a mean of n, then the sum of the numbers is n^2. Likewise m numbers with a mean of m have a sum of m^2. The remaining n – m numbers would have a sum of $n^2 - m^2$ and the mean would be $\frac{n^2-m^2}{n-m} = n + m$ **Ans. n + m**

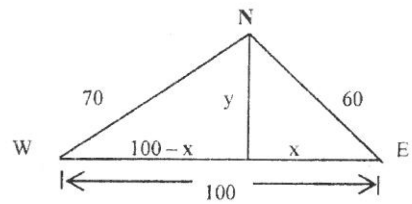
Team



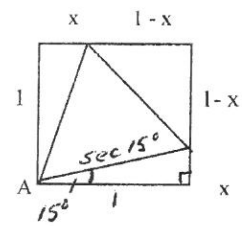
1. In the figure, since QB is bisected, let QP = x. Then PB = x, and AQ = 2x. Thus ΔAQP is a 30-60-90Δ and $m\angle AQP = 60^{\circ}$. Thus $m\angle AQC = 120^{\circ}$. **Ans. 120°**

2. $y^2 - 9x^2 = 36 \Rightarrow \dfrac{y^2}{36} - \dfrac{x^2}{4} = 1$. Endpoints conj. (0, 2), (0, -2) **Ans. (0, 2), (0, -2)**

3. $\sqrt{2}\sin 30^{\circ} = \frac{\sqrt{2}}{2}$. $\cos^{-1}\frac{\sqrt{2}}{2} = 45^{\circ}$ or -45°. $\tan 45^{\circ} = 1$. $\tan -45^{\circ} = -1$. **Ans. 1 or -1**



4. In the figure: $60^2 - x^2 = 70^2 - (100 - x)$. $\Rightarrow$
$3600 - x^2 = 4900 - (10000 - 200x + x^2) \Rightarrow$
$8700 = 200x \Rightarrow x = 43.5$. $y = \sqrt{60^2 - 43.5^2} = 41.3$
Ans. 41 miles



5. Label vertex A where the triangle and square share the same vertex. The other two triangle vertices are equidistant from A. Since the sides of the triangle are congruent, then $1 + x^2 = 2(1 - x)^2 \Rightarrow$
$1 + x^2 = 2 - 4x + 2x^2 \Rightarrow x^2 - 4x - 1 = 0$, $x^2 - 4x + 2 = 3 \Rightarrow x - 2 = \pm\sqrt{3}$.
Thus $x = 2 - \sqrt{3}$. Perimeter $= 3\sqrt{1 + (2-\sqrt{3})^2} = 3.1058$.
Alternate solution: Since $m\angle A$ in the triangle is 60°, then each of the smaller angles adjacent to it are 15°. The hypotenuse or side of the triangle can be found to be sec 15°, since the side of the square is 1. Thus perimeter = 3 sec 15° or 3.1058. **Ans. 3.1058**

6. $\sqrt{2}\sin\theta = \sqrt{2}\cos\theta$, $\sin\theta = \cos\theta$ or $\tan\theta = 1$, thus $\theta = 45^{\circ}$. Using the graphing calculators in polar graphing mode, you will also find that they intersect at (0, 0).
$(1, 45^{\circ}) = \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ **Ans.** $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$**, (0, 0)**

7. Cost per dozen: $\dfrac{y}{\frac{x}{12}} = \dfrac{12y}{x}$. New cost per dozen: $\dfrac{2}{\frac{x+10}{12}} = \dfrac{24}{x+10}$. Since y is an integer and y < 2, then y = 1. So $\dfrac{12}{x} = \dfrac{24}{x+10} + 0.80 \Rightarrow .8x^2 + 20x - 120 = 0 \Rightarrow x^2 + 25x - 150 = 0$
$(x + 30)(x - 5) = 0$. Thus x = 5. **Ans. x = 5, y = 1**

8. $4x^2 + y^2 + 80x - 24y + 508 = 0 \Rightarrow 4(x^2 + 20x + 100) + (y^2 - 24y + 144) = 36$

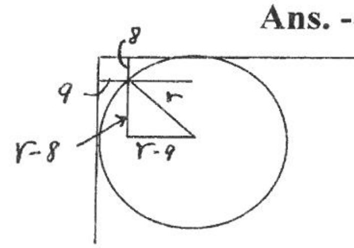
$\frac{(x+10)^2}{9} + \frac{(y-12)^2}{36} = 1$. Vertices are (-10, 12$\pm$6) or (-10, 6), (-10, 18).

$4x^2 + y^2 - 96x + 20y + 576 = 0 \Rightarrow 4(x^2 - 24x + 144) + (y^2 + 20y + 100) = 100$

$\frac{(x-12)^2}{25} + \frac{(y+10)^2}{100} = 1$. Vertices are (12, -10$\pm$ 10) or (12, 0), (12, -20).

The two vertices which are furthest away from each other, one from each ellipse are (-10, 18) and (12,-20). The midpoint of the diameter is (1, -1). The distance from (1, -1) to the point (12, -20) is the radius of the circle: $r = \sqrt{(12-1)^2 + (-20 - -1)^2} = \sqrt{121+361} = \sqrt{482}$. The equation of the circle is $(x-1)^2 + (y+1)^2 = 482$ or $x^2 - 2x + 1 + y^2 + 2x + 1 = 482$. Thus c = -480.

Ans. -480



9. Using the figure at right: $(r-9)^2 + (r-8)^2 = r^2 \Rightarrow$

10.$r^2 - 18r + 81 + r^2 - 16r + 64 = r^2$

$r^2 - 34r + 145 = 0 \Rightarrow (r-29)(r-5) = 0$.

Since 5 is too short, r = 29. **Ans. 29**

Answer Sheet Mar07

Algebraic Fractions and Factoring
1. -1 or y = -1
2. C = 4, D = 2
3. $\frac{-4x^2 + xy - 4x}{x-4}$

Trigonometric Equations and Identities
1. csc x or $\frac{1}{\sin x}$
2. 90^{o} k| k∈I or $\frac{\pi k}{2} | k \in I$
3. either $(-2 - \sqrt{3})$ or $(-\sqrt{7+4\sqrt{3}})$

Circles and Spheres
1. $\frac{\sqrt{3}}{3}$ or $\frac{1}{3}\sqrt{3}$
2. $\frac{\sqrt{3}}{2}$e
3. 13.5

Conics
1. Center (2, -5) radius 6
2. 66
3. $\frac{(x-5)^2}{36} + \frac{(y-2)^2}{32} = 1$

Arithmetic with Statistics
1. 123
2. 14, 15, 19, 21, 21, 24
3. n + m

Team
1. 120 or 120^{o}
2. (2, 0), (−2,0)
3. 1 or -1
4. 41 miles
5. 3.1058
6. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, (0, 0)
7. x = 5, y = 1
8. -480
9. 29 or 29 inches

Meet #5
2007

6 Team Mar07 (You may use Calculators)

3 pts 1. Chord AC is the perpendicular bisector of the radius of a circle with center at Q. Find the degree measure of $\angle AQC$. Ans. ___________

3 pts 2. Find the coordinates of the endpoints of the conjugate axis of the hyperbola $y^2 - 9x^2 = 36$.

Ans. ___________________

3 pts 3. Find the value of $(\tan[\cos^{-1}(\sqrt{2}\sin 30^\circ)])^{-1}$ Ans. ___________________

4 pts 4. Eastville is 100 miles from Westville. Northville is 60 miles from Eastville and 70 miles from Westville. In traveling from Eastville to Westville, what is the closest you come to Northville. Round your answer to the nearest mile.

Ans. ___________

4 pts 5. An equilateral triangle is inscribed in a square of side length 1 so that one vertex of the triangle is located at a vertex of the square. Find the perimeter of the triangle as a decimal number rounded to the nearest ten thousandth.

Ans. ___________

4 pts 6. Using the polar coordinate system, find all intersections of the curves $r = \sqrt{2}\sin\theta$ and $r = \sqrt{2}\cos\theta$. Give the answer(s) in rectangular coordinate form.

Ans. ___________

5 pts 7. A girl entered a store and bought x flowers for y dollars, x and y being integers. When she was about to leave, the clerk said, "If you buy ten more flowers, I will give you all the flowers for two dollars and you will save eighty cents a dozen." Find x and y. Ans. ___________________

5 pts 8. The endpoints of the diameter of a circle are the endpoints of the major axes of the ellipses $4x^2 + y^2 + 80x - 24y + 508 = 0$ and $4x^2 + y^2 - 96x + 20y + 576 = 0$, one from each ellipse which are the furthest distance from each other. If the equation of the circle is in the form $x^2 + y^2 + ax + by + c = 0$, find c.

Ans. ___________

5 pts 9. A circular table is pushed into a corner of a room (touching both walls), so that a point P on the edge of the table is 8 inches from one wall and 9 inches from the other wall. Find the radius of the table in inches.

Ans. ___________